Discrete Applied Mathematics 11 (1985) 307–310
North-Holland

NOTE

A MODIFICATION OF THE ZINOVIEV LOWER BOUND FOR CONSTANT WEIGHT CODES

Iiro HONKALA
Mathematics Department, University of Turku, Turku, Finland

Heikki HÄMÄLÄINEN
Keskipalokka, Finland

Markku KAIKKONEN
Helsinki, Finland

Received 12 April 1984
Revised 3 December 1984

In [3] Zinoviev presented a new method to get lower bounds for constant weight codes. In this note we show that a simple modification of the Zinoviev method gives further improvements.

Let $A(n,2\delta,w)$ denote the maximum number of codewords in any binary code of length n, constant weight w and Hamming distance at least 2δ and let C be a code which attains the bound $A(n,2\delta,w)$. Let L be a k-element subset of the set $E=\{1,2,\dots,n\}$, $0\le k<n$, and c_L the word obtained by deleting those $n-k$ coordinates of the codeword c the indices of which do not belong to L.

Choose an integer g satisfying $0\le g<\min\{w,\delta\}$. Form the set $D(L,i)$, $0\le i\le g$, by changing any $g-i$ 1's to 0's in each element of the set

$$\{c_{E\setminus L} \mid c\in C,\ w(c_L)=i\}.$$

Denote the set $\bigcup_{i=0}^{g} D(L,i)$ by $C(L)$. The code $C(L)$ has length $n-k$ and constant weight $w-g$. If a and b are any two codewords of the original code C with $w(a_L)=i$ and $w(b_L)=j$, $0\le i\le g$, $0\le j\le g$, the distance $d(a,b)$ is at least 2δ and the coordinates of a and b with indices in L differ in at most $i+j$ places. In the construction of the code $C(L)$, $g-i$ and $g-j$ 1's were changed to 0's in $a_{E\setminus L}$ and $b_{E\setminus L}$, respectively, and thus the Hamming distance between any two codewords in $C(L)$ is at least

$$2\delta-(i+j)-(g-i)-(g-j)=2\delta-2g.$$

Let $N(L)$ be the number of codewords in $C(L)$. Evaluate the sum

$$\sum_{\substack{L\subset E\\ |L|=k}} N(L).$$

For each $c \in C$ there are

$$\sum_{i=0}^{g} \binom{w}{i}\binom{n-w}{k-i}$$

k-element subsets L of E with the property that $w(c_L) \leq g$, and for each $c \in C$ and for each choice of L one codeword was formed to the code $C(L)$. Thus

$$\sum_{\substack{L \subset E \\ |L| = k}} N(L) = A(n, 2\delta, w) \sum_{i=0}^{g} \binom{w}{i}\binom{n-w}{k-i}$$

and since $N(L) \leq A(n-k, 2\delta - 2g, w-g)$ for all L we get the following theorem.

Theorem 1. *If* $0 \leq g < \min\{w, \delta\}$ *and* $0 \leq k < n$, *then*

$$A(n-k, 2\delta - 2g, w-g) \geq \frac{\sum_{i=0}^{g} \binom{w}{i}\binom{n-w}{k-i}}{\binom{n}{k}} A(n, 2\delta, w).$$

Using this formula we found several new lower bounds which are given in Table 1 at the end of this note.

If we assume that $k - g < \delta$ (instead of $g < \delta$) and form the set $D(L, i)$, where $g \leq i \leq k$, by changing $i - g$ 0's to 1's in each element of the set

$$\{c_{E-L} \mid c \in C, w(c_L) = i\}$$

and denote $\bigcup_{i=g}^{k} D(L, i)$ by $C(L)$, then the code $C(L)$ has length $n-k$, constant weight $w-g$ and minimum distance at least

$$2\delta - (k-i) - (k-j) - (i-g) - (j-g) = 2(\delta - k + g).$$

In the same way as Theorem 1 we obtain Theorem 1′.

Theorem 1′. *If* $0 \leq g \leq w$, $0 \leq k < n$ *and* $k - g < \delta$, *then*

$$A(n-k, 2(\delta - k + g), w-g) \geq \frac{\sum_{i=g}^{k} \binom{w}{i}\binom{n-w}{k-i}}{\binom{n}{k}} A(n, 2\delta, w).$$

Theorems 1 and 1′ have been proved independently also by Zinoviev [4] and van Pul [2].

The coefficient

$$\frac{\sum_{i=0}^{g} \binom{w}{i}\binom{n-w}{k-i}}{\binom{n}{k}}$$

in Theorem 1 is always smaller than or equal to 1. Theorem 2 presents a case in which the coefficient can be replaced by 1.

Theorem 2. *If* $0<w<n$, *then*

$$A(n-2, d-2, w-1) \geq A(n, d, w).$$

Proof. Let C be a binary constant weight code of length n, minimum distance at least d and weight w, and suppose that C attains the bound $A(n, d, w)$. Denote the set

$$\{c = c_1 c_2 \dots c_n \in C \mid c_{n-1} = c_n = 1\}$$

by E.

Now we form a new constant weight code C' of length n and constant weight $w-1$ by changing each codeword $c = c_1 c_2 \dots c_n$ of the code C in the following way:

(i) If $c \in C - E$, change the last 1 in c to 0.

(ii) If $c \in E$, change the last 0 in c to 1 and change c_{n-1} and c_n to 0's.

We show that the minimum distance of the code C' is at least $d-2$.

Let a and b be any two different codewords of the code C and a' and b' be the codewords of the code C' obtained from a and b using the rules (i)–(ii). If $\{a, b\} \subset C - E$ or $\{a, b\} \subset E$, then clearly $d(a', b') \geq d-2$.

Suppose $a \in C - E$ and $b \in E$, $a = a_1 a_2 \dots a_n$, $b = b_1 b_2 \dots b_n$. Let a_p be the last 1 in the codeword a and b_q the last 0 in b. Then $b_{n-1} = b_n = 1$ and $q \leq n-2$.

Suppose first that $p > q$. If $p > n-2$, then clearly $d(a', b') \geq d-2$. We may assume that $p \leq n-2$. By the definition of q $b_p = 1$. When we change a and b using the rules (i)–(ii), $a_p = 1$ is changed to 0 which increases the distance, $b_q = 0$ is changed to 1 and b_{n-1} and b_n to 0's. Therefore $d(a', b') \geq d-2$.

Second, if $p = q$, then $a_p = 1$ is changed to 0, $b_p = 0$ is changed to 1 and b_{n-1} and b_n are again changed to 0's. Again $d(a', b') \geq d-2$.

Thirdly, if $p < q$, then $a_q = 0$. Now $b_q = 0$ is changed to 1, which increases the distance, and a_p, b_{n-1} and b_n are changed to 0's. So, in all cases $d(a', b') \geq d-2$.

We note that the last two coordinates in each codeword of the code C' are 0's. When we delete them we obtain a constant weight code of length $n-2$, constant weight $w-1$ and minimum distance at least $d-2$. Therefore $A(n-2, d-2, w-1) \geq A(n, d, w)$.

Table 1. In Table 1 we give the improved lower bounds. In the column 'suit. L' we give some lower bounds which have been found using the construction method of Theorem 1 for a suitable choice of the set L. In the five last columns we give the values of g, k, n, 2δ and w to which we apply Theorem 1.

Table 1. Some improved lower bounds

	[1]	[3]	Thm 1	Thm 2	Suit. L	g	k	n	2δ	w
A(22,6,7)	675		682	759		1	2	24	8	8
A(21,6,7)	465		570			1	3	24	8	8
A(20,6,7)	310	320	450			1	4	24	8	8
A(19,6,7)	228	260	338			1	5	24	8	8
A(18,6,7)	160	198	243			1	6	24	8	8
A(17,6,7)	119	141	166			1	7	24	8	8
A(16,6,7)	90	95	108		109	1	8	24	8	8
A(15,6,7)	60		67		69	1	9	24	8	8
A(22,6,11)	1574		1960	2576		1	2	24	8	12
A(21,6,11)	1286		1288			1	3	24	8	12
A(20,6,11)	736		760			1	4	24	8	12
A(19,6,11)	332	360	408			1	5	24	8	12
A(18,6,10)	232		239			1	5	23	8	11
A(23,10,11)	38		46	50		1	1	24	12	12
A(22,10,11)	38			46						

Acknowledgement

The authors wish to thank the referee for useful remarks.

References

[1] R.L. Graham and N.J.A. Sloane, Lower bounds for constant weight codes, IEEE Trans. Inform. Theory 26 (1980) 37–43.

[2] C.L.M. van Pul, A generalization of the Johnson Bound for constant weight codes (unpublished).

[3] V.A. Zinoviev, On a generalization of Johnson upper bound, International Workshop: Convolutional Codes, Multi-User Communication, USSR, Sochi (1983) 206–208.

[4] V.A. Zinoviev, On the generalization of Johnson bound for equal-weight codes, Problems Inform. Trans. 20 (3) (1984) 104–107.